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Note

Chebyshev Series Approximations for the Zeros of the Bessel Functions

1. Introduction

The zeros of the Bessel functions $J_\nu(x)$ have important applications in mathematical physics and applied mathematics. We denote the sth zero of $J_\nu(x)$ by $j_{\nu,s}$. Several approximations, asymptotic expansions or bounds for the zeros of the Bessel functions exist (see [1, 2, 5, 7, 13]. Especially McMahon's expansion for large zeros, (see Olver [7] or Abramowitz and Stegun [1]), Olver's asymptotic expansion for large orders, and Olver's uniform asymptotic expansions (see Olver [7]) are interesting formulas. Unfortunately, they are not applicable when s and ν are small. For that case, which is of frequent occurrence, we have then to use iterative methods for the computation of $j_{\nu,s}$. An excellent computer program, based on an iterative Newton process, is given by Temme [11].

The iterative computation of $j_{\nu,s}$ is very time–consuming, because it requires the evaluation of $J_\nu(x)$. For some applications, we need more efficient formulae. For example, for the estimation of the sth zero, $\xi_s^{(\alpha,n)}$ of the generalized Laguerre polynomial $L_n^{(\alpha)}(x)$, Tricomi [12] has given the following formula

$$\xi_s^{(\alpha,n)} = \frac{j_{\alpha,s}^2}{4k_n}\left[1 + \frac{2(\alpha^2-1)+j_{\alpha,s}^2}{48k_n^2}\right] + O(n^{-5}), \tag{1}$$

where $k_n = n + (\alpha+1)/2$.

A similar formula exists for estimating the zeros of Gegenbauer polynomials [12]. Zeros of these orthogonal polynomials are the abscissae of Gaussian quadrature formulas (Stroud and Secrest [10]). The efficiency of numerical software for constructing Gaussian quadrature formulae is affected by the method for the computation of the abscissae. The most efficient methods are higher order iterative methods, especially when accurate starting values are easily available. Formula (1) gives very accurate approximations for the zeros of $L_n^{(\alpha)}(x)$, (especially for the smallest zeros). But the practical usefulness of (1) depends strongly on the existence of an efficient method for the computation of $j_{\alpha,s}$.

The purpose of this note is to present approximations for $j_{\nu,s}$ in the region of the small ν- and s-values. By using these new approximations, or McMahon's asymptotic expansions, or Olver's uniform asymptotic expansions (depending on the values of ν and s), we are able to calculate $j_{\nu,s}$ to at least 12 decimal figures, in the whole region $\nu \geqslant -1$, $s \geqslant 1$. Olver's uniform asymptotic expansions for $j_{\nu,s}$ are very powerful [7].

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The first four terms yield twelve-figure accuracy when $\nu \geqslant 5$. McMahon's expansion, truncated after 8 terms yields also at least twelve-figure accuracy when $s > 6$ and $\nu < 5$ (the explicit expression for the coefficients of this expansion are given by Olver [7]). For $\nu \geqslant 5$, the range of applicability of these expansions partially overlap.

The gap left by Olver's and McMahon's expansions is partially closed by Chebyshev series expansions presented by Németh [6], which yield fifteen figure accuracy for $0 \leqslant \nu \leqslant \infty$ and $1 \leqslant s \leqslant 10$, and, by rational approximations given by Piessens [8], which are valid only for integer values of ν.

Because for the important region $-1 < \nu < 0$, $1 \leqslant s \leqslant 6$, no accurate expansions or approximations for $j_{\nu,s}$ are known, we present here Chebyshev series approximations for $j_{\nu,s}$, $s = 1, 2, 3, 4, 5$, and 6 as functions of ν on the interval $[-1, 5]$.

TABLE I

Coefficients of the Approximation for $j_{\nu,1}$ (Formula (3))

k	$c_k^{(1)}$
0	5.767950632456
1	0.767665211539
2	−0.086538804759
3	0.020433979038
4	−0.006103761347
5	0.002046841322
6	−0.000734476579
7	0.000275336751
8	−0.000106375704
9	0.000042003336
10	−0.000016858623
11	0.000006852440
12	−0.000002813300
13	0.000001164419
14	−0.000000485189
15	0.000000203309
16	−0.000000085602
17	0.000000036192
18	−0.000000015357
19	0.000000006537
20	−0.000000002791
21	0.000000001194
22	−0.000000000512
23	0.000000000220
24	−0.000000000095
25	0.000000000041
26	−0.000000000018
27	0.000000000008
28	−0.000000000003
29	0.000000000001

TABLE II

Coefficients of the Approximation for $j_{\nu,2}$ (Formula (4), $s = 2$)

k	$c_k^{(2)}$
0	16.526388664614
1	4.209200330779
2	−0.164644722483
3	0.039764618826
4	−0.011799527177
5	0.003893555229
6	−0.001369989689
7	0.000503054700
8	−0.000190381770
9	0.000073681222
10	−0.000029010830
11	0.000011579131
12	−0.000004672877
13	0.000001903082
14	−0.000000781030
15	0.000000322648
16	−0.000000134047
17	0.000000055969
18	−0.000000023472
19	0.000000009882
20	−0.000000004175
21	0.000000001770
22	−0.000000000752
23	0.000000000321
24	−0.000000000137
25	0.000000000059
26	−0.000000000025
27	0.000000000011
28	−0.000000000005
29	0.000000000002

2. Chebyshev Series Approximations for $j_{\nu,s}$

In [9] it is shown that

$$j_{\nu,1} = (\nu + 1)^{1/2} \, [2 + (\nu + 1)/2 + O(\nu + 1)], \qquad \nu \to -1. \tag{2}$$

In order to take account of this asymptotic behaviour for $\nu \to -1, j_{\nu,1}$ is approximated by

$$j_{\nu,1} = (\nu + 1)^{1/2} \sum_{k=0}^{N_1}{}' c_k^{(1)} T_k \left(\frac{\nu - 2}{3}\right), \qquad -1 \leqslant \nu \leqslant 5, \tag{3}$$

where the prime indicates that the first term is taken with factor $\frac{1}{2}$. For the following zeros, we use the approximation

$$j_{\nu,s} = \sum_{k=0}^{N_s}{}' c_k^{(s)} T_k \left(\frac{\nu - 2}{3}\right), \qquad -1 \leqslant \nu \leqslant 5, \, s = 2, 3, 4, 5, 6. \tag{4}$$

TABLE III

Coefficients of the Approximation for $j_{\nu,3}$ (Formula (4), $s = 3$)

k	$c_k^{(3)}$
0	22.987742904346
1	4.317988625384
2	−0.130667664397
3	0.023009510531
4	−0.004987164201
5	0.001204453026
6	−0.000310786051
7	0.000083834770
8	−0.000023343325
9	0.000006655551
10	−0.000001932603
11	0.000000569367
12	−0.000000169722
13	0.000000051084
14	−0.000000015501
15	0.000000004736
16	−0.000000001456
17	0.000000000450
18	−0.000000000140
19	0.000000000043
20	−0.000000000014
21	0.000000000004

TABLE IV

Coefficients of the Approximation for $j_{\nu,4}$ (Formula (4), $s = 4$)

k	$c_k^{(4)}$
0	29.378073011861
1	4.387437455306
2	−0.109469595763
3	0.015359574754
4	−0.002655024938
5	0.000511852711
6	−0.000105522473
7	0.000022761626
8	−0.000005071979
9	0.000001158094
10	−0.000000269480
11	0.000000063657
12	−0.000000015222
13	0.000000003677
14	−0.000000000896
15	0.000000000220
16	−0.000000000054
17	0.000000000013
18	−0.000000000003
19	0.000000000001

TABLE V

Coefficients of the Approximation for $j_{\nu,5}$ (Formula (4), $s = 5$)

k	$c_k^{(5)}$
0	35.733765742756
1	4.435717974422
2	−0.094492317231
3	0.011070071951
4	−0.001598668225
5	0.000257620149
6	−0.000044416219
7	0.000008016197
8	−0.000001495224
9	0.000000285903
10	−0.000000055734
11	0.000000011033
12	−0.000000002212
13	0.000000000448
14	−0.000000000092
15	0.000000000019
16	−0.000000000004

TABLE VI

Coefficients of the Approximation for $j_{\nu,6}$ (Formula (4), $s = 6$)

k	$c_k^{(6)}$
0	42.069568616175
1	4.471319438161
2	−0.083234240394
3	0.008388073020
4	−0.001042443435
5	0.000144611721
6	−0.000021469973
7	0.000003337753
8	−0.000000536428
9	0.000000088402
10	−0.000000014856
11	0.000000002536
12	−0.000000000438
13	0.000000000077
14	−0.000000000014
15	0.000000000002

The coefficients $c_k^{(s)}$ are computed numerically using an algorithm proposed by Gentleman [3]. The values of $j_{\nu,s}$ required for this algorithm, are calculated using an high-order iterative formula [4]. The values of N_s in (3) and (4) are chosen sufficiently large so that the error of the approximation is smaller than 10^{-12}. The values of the coefficients $c_k^{(s)}$ are presented in Tables 1–6.

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R. Piessens

Department of Computer Science, University of Leuven, Celestijnenlaan 200A, B-3030 Leuven (Heverlee), Belgium